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Dynamic response of a non-homogeneous 1D slab under periodic thermal excitation

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Abstract

The periodic heat conduction in a non-homogeneous slab is studied and an approximate analytic solution, valid for weakly non homogeneous slabs, is found. An exact solution for a well defined non-homogeneity is also found and compared to the approximate solution to estimate the accuracy of the result. The solution allows to extend the method of thermal quadrupoles to weakly non-homogeneous 1D-slabs.

1. Introduction

Transient heat conduction in homogeneous materials is a deeply analysed topic, also for the vast consequences in many applied fields, like energy storage [1], laser heating, thermal behaviour of building materials [2], but also for measurement techniques like microcalorimetry [3] or transient techniques for measuring convective heat transfer coefficient [4,5]. Many different techniques were developed to solve transient one-dimensional problems also for the more complex case of composite walls [6]. The case of transient heating of a non-homogeneous materials is more complex and numerical approaches were used to find solutions for a certain class of materials [7,8]. Periodic heat conduction is a relatively less investigated topic, although the implications for many applied fields are significant, for example for building walls and materials [9,10] or again for techniques to measure conductivity and heat capacity like a.c. calorimetry [11] or 3-ω methods [12] and for local convective heat transfer coefficient measurements [13]. The periodic conduction in a 1D homogeneous slab is a relatively simple problem but recently many studies, dealing with it, can be found in the literature [16–19], particularly related to the hyperbolic version of the heat equation, as the wave like behaviour can be better evidenced in a simple geometry. However, the extension of the transient conduction analysis to non-homogeneous material is not straightforward, and fewer works can be found on this subject [14,15], whereas periodic conduction in non-homogeneous material does not seems to have been sufficiently considered, although non-homogeneous material are becoming everyday more widespread in many engineering applications. The paper presents a solution of the periodic conduction in a non-homogeneous 1D slab under the condition of weak non-homogeneity (i.e. limited variation of the material properties with position) and an assessment of the accuracy of the method by comparison with an exact solution. The solution allows a relatively simple extension of the thermal quadrupole method to this class of materials.

2. Basic equations

Consider a 1D slab made of a non-homogeneous material, the energy equation can then be written as

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} \tag{1}$$

and making use of Fourier law: $q = -k\frac{\partial T}{\partial x}$ we obtain:

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$$d^2\frac{\partial T}{\partial t} = \gamma(\xi)\frac{\partial T}{\partial \xi} + \alpha(\xi)\frac{\partial^2 T}{\partial \xi^2} \quad (2)$$

where $\xi = \frac{x}{d}$, $\gamma(\xi) = \frac{1}{\rho c}\frac{\partial k}{\partial \xi}$; and $\alpha(\xi) = \frac{k}{\rho c}$. The temperature and heat flux fields can be split into time average (T_a, q_a) and fluctuating $(\widetilde{T}, \tilde{q})$ components. Introducing the Fourier transform of the latter:

$$T(x,t) = T_a(x) + \int_{-\infty}^{+\infty} S(\omega, x)e^{i\omega t}d\omega;$$

$$q(x,t) = q_a(x) + \int_{-\infty}^{+\infty} Q(\omega, x)e^{i\omega t}d\omega$$

Eq. (2) becomes:

$$i\omega d^2 S = \gamma(\xi)S' + \alpha(\xi)S'' \quad (3)$$

$$0 = \gamma(\xi)T_a' + \alpha(\xi)T_a'' \quad (4)$$

where apex means derivation respect to ξ. The case of homogeneous slab is found setting $\gamma = 0$ and $\alpha = \alpha_0 =$ const. and Eq. (3) becomes:

$$S_H'' - \beta^2 S_H = 0$$

with $\beta = \sqrt{\frac{i\omega d^2}{\alpha_0}}$, the solution is easily found under the more convenient form:

$$S_H = S_{H,0}\cosh(\beta\xi) + \frac{S_{H,0}'}{\beta}\sinh(\beta\xi) \quad (5)$$

with $S_{H,0} = S_H(0)$ and $S_{H,0}' = S_H'(0)$. The general solution of Eq. (4) is easily found by double integration:

$$T_a = A + B\int_0^{\xi} e^{\int_0^{\zeta} -\frac{\gamma(\chi)}{\alpha(\chi)}d\chi} d\zeta.$$

3. The weakly non-homogeneous slab

Consider a material characterised by variations of its properties with position that are small compared to the average value, i.e.:

$$k = k_0 + \kappa(\xi); \quad \text{with} \quad |\kappa(\xi)| \ll |k_0|$$

$$|\kappa'(\xi)| = |k'(\xi)| \ll |k(\xi)|$$

$$\alpha = \alpha_0 + a(\xi); \quad \text{with} \quad |a(\xi)| \ll |\alpha_0|$$

from which it follows: $|\frac{\gamma}{\alpha}| = |\frac{k'}{k}| \ll 1$. Suppose now that the solution of Eq. (3) can be written as a perturbation of the solution S_H valid for homogeneous materials (Eq. (5)), i.e.

$$S = S_H + s \quad \text{with} \quad s = o(S_H)$$

then substituting into Eq. (3) and preserving only the terms up to the first order:

$$s'' - \frac{i\omega d^2}{\alpha_0}s = -\frac{\gamma(\xi)}{\alpha_0}S_H' - \frac{a(\xi)}{\alpha_0}S_H'' = m(\xi) \quad (6)$$

where

$$m(\xi) = S_{H0}m_1(\xi) + \frac{S_{H0}'}{\beta}m_2(\xi) \quad (7)$$

$$m_1(\xi) = -\left\{\frac{\gamma(\xi)}{\alpha_0}\beta\sinh(\beta\xi) + \frac{a(\xi)}{\alpha_0}\beta^2\cosh(\beta\xi)\right\};$$

$$m_2(\xi) = -\left\{\frac{\gamma(\xi)}{\alpha_0}\beta\cosh(\beta\xi) + \frac{a(\xi)}{\alpha_0}\beta^2\sinh(\beta\xi)\right\}$$

are known functions. The solution of (6) is then:

$$s = s_0\cosh(\beta\xi) + \frac{1}{\beta}s_0'\sinh(\beta\xi) - \frac{1}{\beta}\int_0^{\xi} m(z)\sinh[\beta(z-\xi)]dz$$

and choosing $s(0) = 0$, $s'(0) = 0$, we obtain:

$$S = S_0\left\{\cosh(\beta\xi) - \frac{1}{\beta}\int_0^{\xi} m_1(z)\sinh[\beta(z-\xi)]dz\right\} + \frac{S_0'}{\beta}\left\{\sinh(\beta\xi) - \frac{1}{\beta}\int_0^{\xi} m_2(z)\sinh[\beta(z-\xi)]dz\right\} \quad (8)$$

with $S_0 = S(0)$ and $S_0' = S'(0)$. The linearisation that led to Eq. (6) introduces approximation whose entity depends on the magnitude of the slab thermal properties variations. To assess the validity of the approximation a comparison to an exact solution of Eq. (3), for a particular distribution of the slab properties, is needed.

4. An exact solution

Consider the following particular form for the dependence on position of the material properties:

$$\rho c = \text{const}; \; k = K(\xi_0 \pm \xi)^2 \quad (9)$$

Eq. (3) becomes

$$(\xi_0 \pm \xi)^2 S'' + 2(\xi \pm \xi_0)S' - i\frac{\rho c\omega d^2}{K}S = 0$$

and after the change of variable: $\eta = (\xi \pm \xi_0)$, and defining $\lambda_0 = \frac{1}{2}\sqrt{1 + i\frac{4\omega\rho c d^2}{K}}$, an Euler equation is promptly recovered:

$$\eta^2 S'' + 2\eta S' + \left(\frac{1}{4} - \lambda_0^2\right)S = 0 \quad (10)$$

where now the apex means derivation respect to η. The solution of this equation can be written in the convenient form:

$$S = \frac{1}{\lambda_0}\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)^{-\frac{1}{2}}\left\{S_0\left[\frac{1}{2}\sinh(z) + \lambda_0\cosh(z)\right] \pm S_0'\xi_0\sinh(z)\right\} \quad (11)$$

with

$$z = \lambda_0 \ln\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)$$

and again: $S(0) = S_0$, $S'(0) = S_0'$. The exact analytic solution can now be used to assess the validity of the approximate solution (8).

5. Assessment of the perturbative solution

Consider again the material described by (9) with the further condition $\xi_0 \gg 1$, then, after defining: $k_0 = K\xi_0^2$ we obtain:

$$\kappa(\xi) = k - k_0 = K\xi^2 \pm 2K\xi_0\xi$$
$$k'(x) = 2K(\xi \pm \xi_0)$$

and the condition: $|k'| \ll |k|$ yields:

$$2 \ll |(\xi \pm \xi_0)|$$

that is obviously verified as $0 \leqslant \xi \leqslant 1$ and $\xi_0 \gg 1$. Moreover

$$\alpha_0 = \frac{k_0}{\rho c} = \frac{K\xi_0^2}{\rho c}; \quad \frac{\gamma(\xi)}{\alpha_0} = \frac{2(\xi \pm \xi_0)}{\xi_0^2}; \quad \frac{a(\xi)}{\alpha_0} = \frac{\xi^2 \pm 2\xi_0\xi}{\xi_0^2}$$

The weakly non-homogeneous approximation holds and the functions m_1 and m_2 (see Eq. (7)) now become:

$$m_1(\xi) = -\left\{\frac{2(\xi \pm \xi_0)}{\xi_0^2}\beta \sinh(\beta\xi) + \frac{\xi^2 \pm 2\xi_0\xi}{\xi_0^2}\beta^2 \cosh(\beta\xi)\right\};$$
$$m_2(\xi) = -\left\{\frac{2(\xi \pm \xi_0)}{\xi_0^2}\beta \cosh(\beta\xi) + \frac{\xi^2 \pm 2\xi_0\xi}{\xi_0^2}\beta^2 \sinh(\beta\xi)\right\}$$

with $\beta = \sqrt{\mathrm{i}\frac{\omega d^2}{\alpha_0}}$ and the following integrals:

$$I_1(\xi) = \int_0^{\xi} m_1(z) \sinh(\beta[z - x])\,\mathrm{d}z$$
$$I_2(\xi) = \int_0^{\xi} m_2(z) \sinh(\beta[z - x])\,\mathrm{d}z$$

must be evaluated to find the solution in an explicit form. In Appendix, the integrals are calculated, finding:

$$I_1(\xi) = -\frac{\beta}{\xi_0^2}\{M_1 \sinh(\beta\xi) + N_1 \cosh(\beta\xi)\}$$
$$I_2(\xi) = -\frac{\beta}{\xi_0^2}\{M_2 \sinh(\beta\xi) + N_2 \cosh(\beta\xi)\}$$

with

$$M_1 = \pm\frac{\xi_0}{2\beta} + \frac{1}{4\beta}\xi \mp \frac{\xi_0}{2}\beta\xi^2 - \frac{\beta}{6}\xi^3; \quad N_1 = \mp\frac{\xi_0\xi}{2} - \frac{1}{4}\xi^2$$
$$M_2 = -\frac{1}{4\beta^2} \mp \frac{\xi_0}{2}\xi - \frac{1}{4}\xi^2; \quad N_2 = \frac{1}{4\beta}\xi \mp \frac{\xi_0\beta}{2}\xi^2 - \frac{\beta}{6}\xi^3$$

The approximate solution then becomes:

$$S = S_{H0}\left\{\left(1 + \frac{N_1(\xi)}{\xi_0^2}\right)\cosh(\beta\xi) + \frac{M_1(\xi)}{\xi_0^2}\sinh(\beta\xi)\right\} + \frac{S'_{H0}}{\beta}\left\{\sinh(\beta\xi)\left[1 + \frac{M_2(\xi)}{\xi_0^2}\right] + \frac{N_2(\xi)}{\xi_0^2}\cosh(\beta\xi)\right\} \tag{12}$$

The exact (11) and approximate (12) solutions can now be compared by comparing the functions:

$$F_{1,\mathrm{ap}} = \left\{\left(1 + \frac{N_1(\xi)}{\xi_0^2}\right)\cosh(\beta\xi) + \frac{M_1(\xi)}{\xi_0^2}\sinh(\beta\xi)\right\}$$
$$F_{2,\mathrm{ap}} = \frac{1}{\beta}\left\{\sinh(\beta\xi)\left[1 + \frac{M_2(\xi)}{\xi_0^2}\right] + \frac{N_2(\xi)}{\xi_0^2}\cosh(\beta\xi)\right\} \tag{13}$$

with the functions:

$$F_{1,\mathrm{ex}} = \frac{1}{\lambda_0}\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)^{-\frac{1}{2}}\left[\frac{1}{2}\sinh\left(\lambda_0 \ln\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)\right) + \lambda_0 \cosh\left(\lambda_0 \ln\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)\right)\right]$$
$$F_{2,\mathrm{ex}} = \pm\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)^{-\frac{1}{2}}\frac{\xi_0}{\lambda_0}\sinh\left(\lambda_0 \ln\left(\frac{\xi_0 \pm \xi}{\xi_0}\right)\right) \tag{14}$$

respectively, with now $\lambda_0 = \frac{1}{2}\sqrt{1 + 4\beta^2\xi_0^2}$.

Fig. 1 reports a comparison for a fixed value of ω, having introduced the non-homogeneity magnitude ε defined as

$$\varepsilon = \frac{k_{\max} - k_{\min}}{k_0} = \frac{|(1 \pm 2\xi_0)|}{\xi_0^2}$$

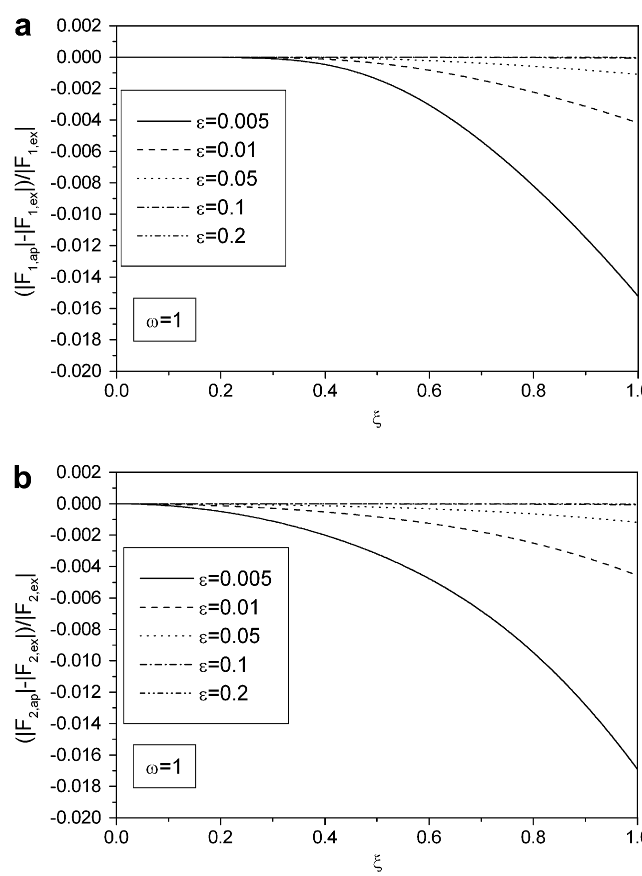

Fig. 1. Relative discrepancies between the approximated and the exact values of: (a) the function F_1 and (b) the function F_2 versus the position along the slab, for $\omega = 1$.

It appears that the maximum discrepancy occurs at $\xi = 1$, thus it is interesting to compare the two solutions (by comparing the functions (13) and (14)) at the fixed position $\xi = 1$ for different values of the exciting frequency. To this end, observe that by introducing the parameter $\tilde{\varepsilon} = \xi_0^{-1} = \varepsilon + o(\varepsilon)$, the coefficients of the hyperbolic functions in $F_{j,a}$ evaluated for $\xi = 1$ can be expressed as polynomials of second order in $\tilde{\varepsilon}$:

$$C_{11,a} = 1 + \frac{N_1}{\xi_0^2} = 1 \mp \frac{\tilde{\varepsilon}}{2} - \frac{\tilde{\varepsilon}^2}{4}$$
$$C_{12,a} = \frac{M_1}{\xi_0^2} = \pm\frac{\tilde{\varepsilon}}{2}\left(\frac{1}{\beta} - \beta\right) + \frac{\tilde{\varepsilon}^2}{2}\left(\frac{1}{2\beta} - \frac{\beta}{3}\right)$$
$$C_{21,a} = \frac{1}{\beta}\frac{N_2}{\xi_0^2} = \mp\frac{\tilde{\varepsilon}}{2} + \frac{\tilde{\varepsilon}^2}{2}\left(\frac{1}{2\beta^2} - \frac{1}{3}\right)$$
$$C_{22,a} = \frac{1}{\beta}\left(1 + \frac{M_2}{\xi_0^2}\right) = \frac{1}{\beta}\left[1 \mp \frac{\tilde{\varepsilon}}{2} - \frac{\tilde{\varepsilon}^2}{4}\left(\frac{1}{\beta^2} + 1\right)\right]$$

Moreover, defining: $\varphi = \lambda_0 \ln\left(\frac{\xi_0 \pm 1}{\xi_0}\right) - \beta$ the functions $F_{j,e}$ can be written (for $\xi = 1$) as

$$F_{1,\mathrm{ex}} = \left[(1 \pm \tilde{\varepsilon})^{-\frac{1}{2}}\left(\frac{1}{2\lambda_0}\sinh(\varphi) + \cosh(\varphi)\right)\cosh(\beta) + (1 \pm \tilde{\varepsilon})^{-\frac{1}{2}}\left(\frac{1}{2\lambda_0}\cosh(\varphi) + \sinh(\varphi)\right)\sinh(\beta)\right]$$
$$F_{2,\mathrm{ex}} = \pm\left(\frac{(1 \pm \tilde{\varepsilon})^{-\frac{1}{2}}}{\tilde{\varepsilon}\lambda_0}\sinh(\varphi)\cosh(\beta) + \frac{(1 \pm \tilde{\varepsilon})^{-\frac{1}{2}}}{\tilde{\varepsilon}\lambda_0}\cosh(\varphi)\sinh(\beta)\right)$$

and the corresponding coefficients of the hyperbolic functions are (up to Second order in $\tilde{\varepsilon}$):

$$\begin{aligned}
C_{11,e} &= 1 \mp \frac{\tilde{\varepsilon}}{2} + (1 + \beta^2)\frac{\tilde{\varepsilon}^2}{8} + o(\tilde{\varepsilon}^2) \\
C_{12,e} &= \pm\frac{\tilde{\varepsilon}}{2}\left(\frac{1}{\beta} - \beta\right) + \frac{\tilde{\varepsilon}^2}{4}\left(\frac{7\beta}{3} - \frac{1}{2\beta}\right) + o(\tilde{\varepsilon}^2) \\
C_{21,e} &= \mp\frac{\tilde{\varepsilon}}{2} + \frac{\tilde{\varepsilon}^2}{4}\left(\frac{7\beta^2}{3\beta^2} + \frac{1}{2\beta^2}\right) + o(\tilde{\varepsilon}^2) \\
C_{22,e} &= \frac{1}{\beta}\left(1 \mp \frac{\tilde{\varepsilon}}{2} + \frac{1}{8}\tilde{\varepsilon}^2\frac{(\beta^4 + 3\beta^2 - 1)}{\beta^2} + o(\tilde{\varepsilon}^2)\right)
\end{aligned} \tag{15}$$

a comparison with those relative to the approximate solution shows that the error is of order $\tilde{\varepsilon}^2$, i.e. of order ε^2

$$\begin{aligned}
\Delta C_{11} &= C_{11,a} - C_{11,e} = -\frac{\tilde{\varepsilon}^2}{8}(3 + \beta^2) + o(\tilde{\varepsilon}^2) \\
\Delta C_{12} &= C_{12,a} - C_{12,e} = \frac{3\tilde{\varepsilon}^2}{4}\left[\frac{1}{2\beta} - \beta\right] + o(\tilde{\varepsilon}^2) \\
\Delta C_{21} &= C_{21,a} - C_{21,e} = \frac{\tilde{\varepsilon}^2}{4}\left[\frac{1}{2\beta^2} - 3\right] + o(\tilde{\varepsilon}^2) \\
\Delta C_{22} &= C_{22,a} - C_{22,e} = -\frac{\tilde{\varepsilon}^2}{4}\left(\frac{\beta^2 + 4}{2\beta}\right) + o(\tilde{\varepsilon}^2)
\end{aligned} \tag{16}$$

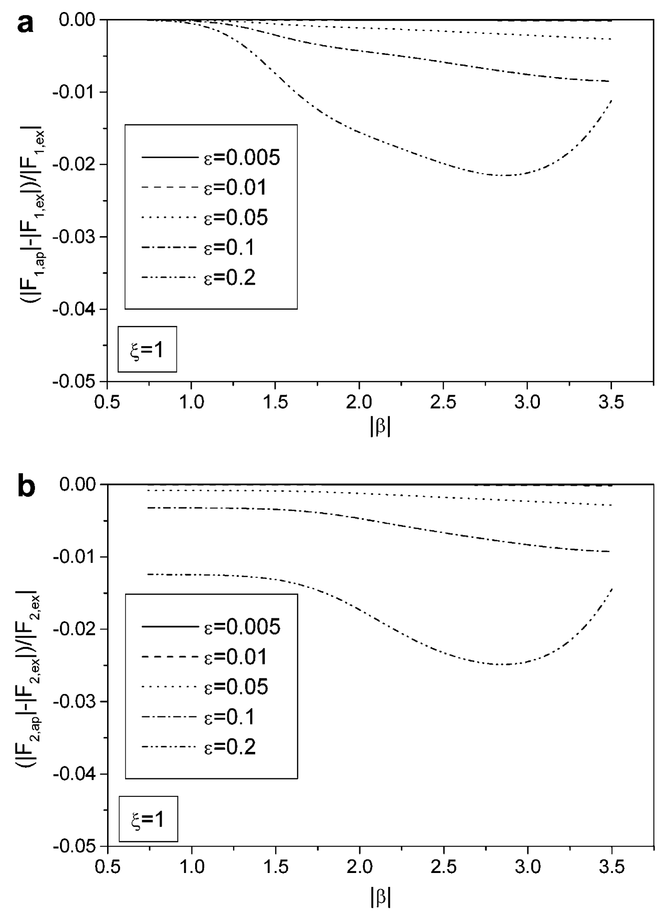

Fig. 2. Relative discrepancies between the approximated and the exact values of: (a) the function F_1 and (b) the function F_2 versus non-dimensional frequency (β) for $\xi = 1$.

Fig. 2 shows the comparison of the two solutions, the discrepancy between the approximate and the exact one remains limited also for relatively large (0.2) values of ε. However, relative errors ($\Delta C_{jk}/C_{jk,e}$) are expected to increase for very large and very small values of the frequency ω, as it stems from Eqs. (15) and (16).

6. The thermal quadrupole approach for the non-homogeneous slab

The thermal quadrupole approach [20] gives a compact and elegant way to solve transient problems for 1D homogeneous slabs and the application to multilayered slabs is straightforward. The previous results allow to extend the thermal quadrupole approach also to 1D non-homogeneous materials. Defining

$$\Psi(x, \omega) = \begin{bmatrix} S \\ Q \end{bmatrix}$$

the following relation, that is the basis of the thermal quadrupole method, holds for a homogeneous slab:

$$\Psi(d, \omega) = \mathbf{M}_h \Psi(0, \omega)$$

where

$$\mathbf{M}_h = \begin{bmatrix} \cosh(\beta\xi) & -\frac{1}{k\beta}\sinh(\beta\xi) \\ -k\beta\sinh(\beta\xi) & \cosh(\beta\xi) \end{bmatrix}$$

For the non-homogeneous slab a similar equation can now be written, in fact:

$$S = S_0\left\{\cosh(\beta\xi) - \frac{1}{\beta}\int_0^\xi m_1(z)\sinh[\beta(z-\xi)]\,dz\right\} - \frac{Q_0}{k\beta}\left\{\sinh(\beta\xi) - \frac{1}{\beta}\int_0^\xi m_2(z)\sinh[\beta(z-\xi)]\,dz\right\}$$

$$Q = S_0\left\{-k\beta\sinh(\beta\xi) - k\int_0^\xi m_1(z)\cosh[\beta(z-\xi)]\,dz\right\} + Q_0\left\{\cosh(\beta\xi) + \frac{1}{\beta}\int_0^\xi m_2(z)\cosh[\beta(z-\xi)]\,dz\right\}$$

and then defining:

$$\mathbf{M}_{nh} = \begin{bmatrix} \cosh(\beta\xi) & -\frac{1}{k\beta}\sinh(\beta\xi) \\ -k\beta\sinh(\beta\xi) & \cosh(\beta\xi) \end{bmatrix} + \begin{bmatrix} -\frac{1}{\beta}\int_0^\xi m_1(z)\sinh[\beta(z-\xi)]\,dz & \frac{1}{k\beta^2}\int_0^\xi m_2(z)\sinh[\beta(z-\xi)]\,dz \\ -k\int_0^\xi m_1(z)\cosh[\beta(z-\xi)]\,dz & \frac{1}{\beta}\int_0^\xi m_2(z)\cosh[\beta(z-\xi)]\,dz \end{bmatrix} = \mathbf{M}_h + \begin{bmatrix} -\frac{1}{\beta}I_1 & \frac{1}{k\beta^2}I_2 \\ -kI_3 & \frac{1}{\beta}I_4 \end{bmatrix}$$

the following equation is obtained:

$$\Psi(d,\omega) = \mathbf{M}_{nh}\Psi(0,\omega)$$

If the functions $\gamma(\xi)$ and $\alpha(\xi)$ can be expressed as polynomial in ξ, then the integrals I_j can be written in a closed form (see Appendix). These results allow to extend the use of the thermal quadrupole method to 1-d weakly non-homogeneous slabs.

7. Conclusions

The thermal response of a weakly non-homogeneous slab to periodic thermal excitation was analysed through a Fourier transform approach and an analytic solution of the transformed problem, valid in the limit of small non-homogeneity, was found. The solution can be written in closed form whenever the dependence of the wall thermal characteristics on the position can be represented in polynomial form. To assess the validity of the approximation, a comparison with an exact solution of the transformed problem is reported, showing that accuracy is of second order in the non-homogeneity parameter ε. The solution allows also to extend the method of thermal quadrupoles to the weakly non-homogeneous slab thus extending the capability of the method.

Appendix

Defining the integrals:

$$I_n^\pm = \int_0^x z^n \exp(\pm 2\beta z)\,dz$$

and using partial integration, the following recurrence relation is found:

$$I_n^\pm = \frac{e^{\pm 2\beta x}}{\pm 2\beta}x^n - \frac{n}{\pm 2\beta}I_{n-1}^\pm$$

where:

$$I_0^\pm = \int_0^x \exp(\pm 2\beta z)\,dz = \frac{e^{\pm 2\beta x} - 1}{\pm 2\beta}$$

and the closed form:

$$I_n^\pm = \frac{e^{\pm 2\beta x}}{\pm 2\beta}\sum_{k=0}^{n} x^k \frac{(-1)^{n-k} n!}{k!(\pm 2\beta)^{n-k}} + (-1)^{n+1}\frac{n!}{(\pm 2\beta)^{n+1}} \tag{17}$$

is promptly found. Consider now the four integrals:

$$I_{ss}^n = \int_0^x z^n \sinh(\beta z)\sinh(\beta(z-x))\,dz$$
$$I_{cs}^n = \int_0^x z^n \cosh(\beta z)\sinh(\beta(z-x))\,dz$$
$$I_{sc} = \int_0^x z^n \sinh(\beta z)\cosh(\beta(z-x))\,dz$$
$$I_{cc} = \int_0^x z^n \cosh(\beta z)\cosh(\beta(z-x))\,dz$$

a simple calculation gives:

$$I_{ss}^n = A_+ - \frac{x^{n+1}}{2(n+1)}\cosh(\beta x);\quad I_{cs}^n = A_- - \frac{x^{n+1}}{2(n+1)}\sinh(\beta x);$$
$$I_{sc}^n = A_- + \frac{x^{n+1}}{2(n+1)}\sinh(\beta x);\quad I_{cc}^n = A_+ + \frac{x^{n+1}}{2(n+1)}\cosh(\beta x) \tag{18}$$

where

$$A_\pm = \frac{1}{4}[e^{-\beta x}I_n^+ \pm e^{\beta x}I_n^-]$$

and using Eq. (17), after few manipulations, yields:

$$A_+ = \left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}} x^{n-2s}a_{n,n-2s}\right)\sinh(\beta x) + \left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}} x^{n-2s+1}a_{n,n-2s+1}\right)\cosh(\beta x) + \frac{n!}{2(2\beta)^{n+1}}\begin{cases}\sinh(\beta x) & n \text{ even}\\ \cosh(\beta x) & n \text{ odd}\end{cases}$$

$$A_- = \left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}} x^{2s}a_{n,2s}\right)\cosh(\beta x) + \left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}} x^{2s+1}a_{n,2s+1}\right)\sinh(\beta x) - \frac{n!}{2(2\beta)^{n+1}}\begin{cases}\sinh(\beta x) & n \text{ odd}\\ \cosh(\beta x) & n \text{ even}\end{cases}$$

Substituting back into Eq. (18) yields the following result:

$$I_{ss}^n = Y_s^n\sinh(\beta\xi) + Z_s^n\cosh(\beta\xi);\quad I_{cs}^n = Y_c^n\sinh(\beta\xi) + Z_c^n\cosh(\beta\xi)$$
$$I_{sc} = K_s^n\sinh(\beta\xi) + P_s^n\cosh(\beta\xi);\quad I_{cc} = K_c^n\sinh(\beta\xi) + P_c^n\cosh(\beta\xi)$$

where

$$Y_s^n=\left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}}x^{n-2s}a_{n,n-2s}+\frac{1+(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right);$$

$$Z_s^n=\left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}}x^{n-2s+1}a_{n,n-2s+1}-\frac{x^{n+1}}{2(n+1)}+\frac{1-(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right)$$

$$Y_c^n=\left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}}x^{n-2s+1}a_{n,n-2s+1}-\frac{x^{n+1}}{2(n+1)}-\frac{1-(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right);$$

$$Z_c^n=\left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}}x^{n-2s}a_{n,n-2s}-\frac{1+(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right)$$

$$K_s^n=\left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}}x^{n-2s+1}a_{n,n-2s+1}+\frac{x^{n+1}}{2(n+1)}-\frac{1-(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right);$$

$$P_s^n=\left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}}x^{n-2s}a_{n,n-2s}-\frac{1+(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right)$$

$$K_c^n=\left(\frac{1}{4\beta}\sum_{s=0}^{\frac{n}{2}}x^{n-2s}a_{n,n-2s}+\frac{1+(-1)^n}{2}\frac{n!}{2(2\beta)^{n+1}}\right);$$

$$P_c^n=\left(-\frac{1}{4\beta}\sum_{s=0}^{\frac{n-1}{2}}x^{n-2s+1}a_{n,n-2s+1}+\frac{x^{n+1}}{2(n+1)}+\frac{n!}{2(2\beta)^{n+1}}\frac{1-(-1)^n}{2}\right)$$

and $a_{n,k}=\frac{(-1)^n n!}{k!(2\beta)^{n-k}}$.

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